# Mimetic Preconditioners for Mixed Discretizations of the Diffusion Equation

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## Class of problems

We consider the 2D steady-state linear diffusion

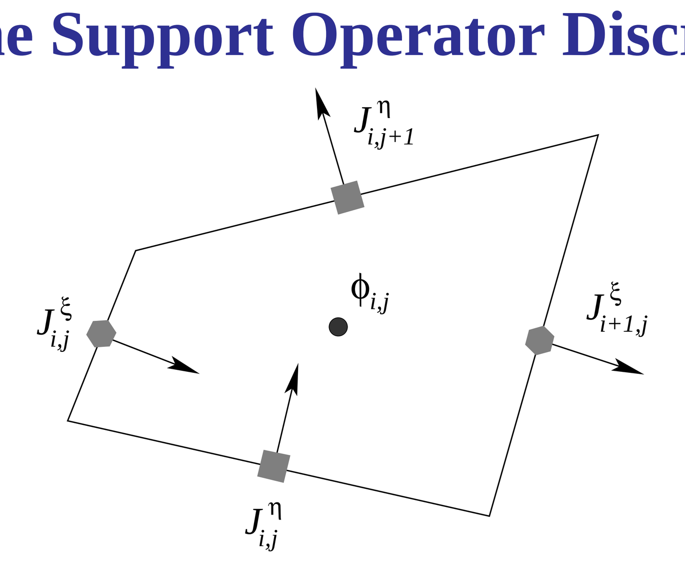
$$\begin{aligned} \nabla \cdot \mathbf{J} &= Q(x,y) \\ \mathbf{J} &= -D(x,y)\nabla\phi \end{aligned}$$

subject to the Dirichlet boundary condition $\phi(x,y) = g(x,y)$.

*Key issues influencing the discretization and the linear solver include:*

- ➢ $D(x,y)$ is defined on, and possibly discontinuous on, a very fine scale
- ➢ coarse-scale view of the fine-scale structure may be anisotropic
- ➢ although the grid is logically rectangular, it may be severly distorted.

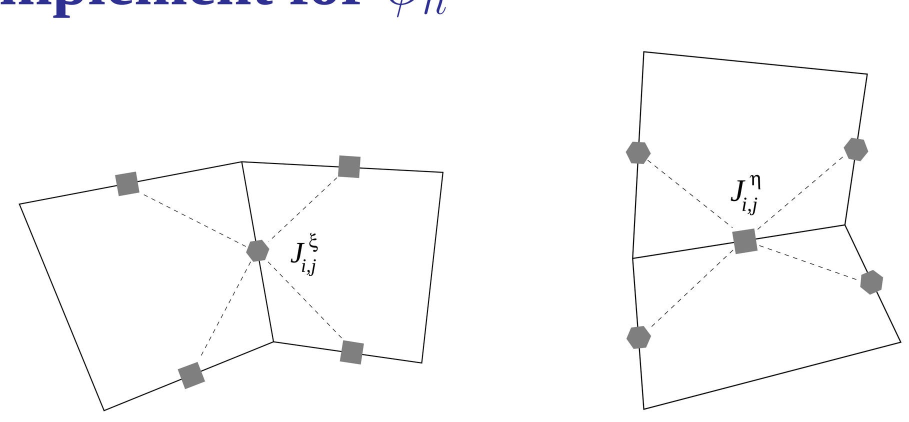
## The Support Operator Discretization




$$\begin{array}{l} \textit{(current)} \\ \textit{(balance)} \end{array} \begin{bmatrix} \mathcal{A} & -\mathcal{D}^\dagger\mathcal{M} \\ -\mathcal{M}\mathcal{D} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{J}_h \\ \boldsymbol{\phi}_h \end{bmatrix}$$

- ➢ Natural discretization of the divergence, $\mathcal{D}$
- ➢ Derive the corresponding gradient, $-\mathcal{D}^\dagger\mathcal{M}$
- ➢ Generate a symmetric, but indefinite, linear system.

## The Schur Complement for $\phi_h$




Stencil schematic for the sparsity structure of $\mathcal{A}$.

Properties of the Schur complement for $\phi_h$:

- ➢ $\mathcal{S}_\phi = \left[\mathcal{M}\mathcal{D}\right]\left(\mathcal{A}\right)^{-1}\left[\mathcal{D}^\dagger\mathcal{M}\right]$
- ➢ $\mathcal{S}_\phi$ is a symmetric positive definite matrix.
- ➢ $\mathcal{S}_\phi$ is a global matrix, but with sparse factors.
- $\Rightarrow$ the matrix-vector product needed by CG is possible.

## Mimetic Preconditioner I: A Simple Lumping of $\mathcal{A}$

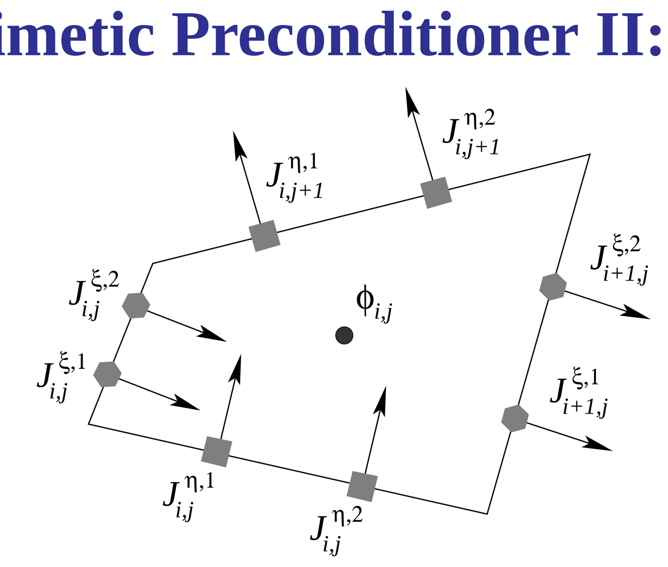
*Diagonal approximation of the flux matrix*:

$$\begin{bmatrix} \mathcal{A} & -\mathcal{D}^\dagger\mathcal{M} \\ -\mathcal{M}\mathcal{D} & 0 \end{bmatrix} \text{ lumping} \Longrightarrow \begin{bmatrix} \widehat{\mathcal{A}} & -\mathcal{D}^\dagger\mathcal{M} \\ -\mathcal{M}\mathcal{D} & 0 \end{bmatrix}$$

Properties of the resulting Schur complement for $\phi_h$:

- ➢ $\mathcal{S}^D_\phi = \left[\mathcal{M}\mathcal{D}\right]\left(\widehat{\mathcal{A}}\right)^{-1}\left[\mathcal{D}^\dagger\mathcal{M}\right]$
- ➢ $\mathcal{S}^D_\phi$ is a symmetric positive definite matrix.
- ➢ $\mathcal{S}^D_\phi$ has a 5-point cell-based stencil
- ➢ Approximately invert $\mathcal{S}^D_\phi$ with a single V-cycle of multigrid.

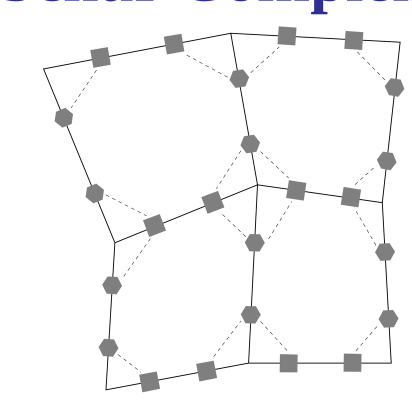
## Mimetic Preconditioner II: Two-Flux Support Operator




$$\begin{array}{l} \textit{(current)} \\ \textit{(balance)} \end{array} \begin{bmatrix} \widetilde{\mathcal{A}} & -\widetilde{\mathcal{D}}^\dagger\mathcal{M} \\ -\mathcal{M}\widetilde{\mathcal{D}} & 0 \end{bmatrix} \begin{bmatrix} \widetilde{\boldsymbol{J}}_h \\ \boldsymbol{\phi}_h \end{bmatrix}$$

- ➢ Natural discretization of the divergence, $\widetilde{\mathcal{D}}$
- ➢ Derive the corresponding gradient, $-\widetilde{\mathcal{D}}^\dagger\mathcal{M}$
- ➢ Generate a symmetric, but indefinite, linear system.

## The Schur Complement for $\phi_h$



Schematic for the structure of $\widetilde{\mathcal{A}}$

$\Longrightarrow$ Order $\widetilde{\boldsymbol{J}}_h$ around vertices, then $\widetilde{\mathcal{A}}$ has a block diagonal structure, with 4x4 blocks.

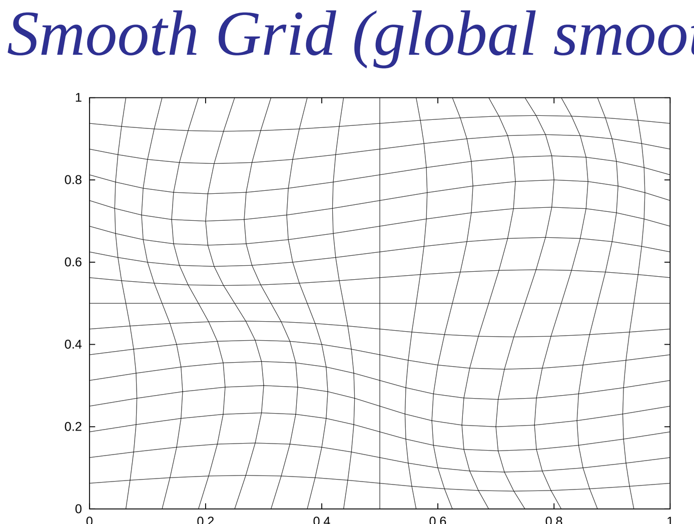
Properties of the resulting Schur complement for $\phi_h$:

- ➢ $\mathcal{S}^{TF}_\phi = \left[\mathcal{M}\widetilde{\mathcal{D}}\right]\left(\widetilde{\mathcal{A}}\right)^{-1}\left[\widetilde{\mathcal{D}}^\dagger\mathcal{M}\right]$
- ➢ $\mathcal{S}^{TF}_\phi$ is a symmetric positive definite matrix.
- ➢ $\mathcal{S}^{TF}_\phi$ has a 9-point cell-based stencil (second order on smooth grids)
- ➢ Approximately invert $\mathcal{S}^{TF}_\phi$ with a single V-cycle of multigrid.

## Examples:

- ➢ $D(x,y) = 1$, Dirichlet boundary conditions, polynomial solution.
  - ➠ the challenge arises from the distorted grid.
- ➢ Iteration counts are for CG on the cell-based system, $\mathcal{S}_\phi$
  - ➠ $CG(\mathcal{S}_\phi)$ denotes only diagonal scaling of $\mathcal{S}_\phi$
  - ➠ $CG(\mathcal{S}^D_\phi)$ denotes the lumped preconditioner
  - ➠ $CG(\mathcal{S}^{TF}_\phi)$ denotes two-flux preconditioner
- ➢ Convergence criteria is a relative residual in the $l_2$ norm of $10^{-6}$.
- ➢ A single V(1,1) cycle of BoxMG inverts the cell-based preconditioner.

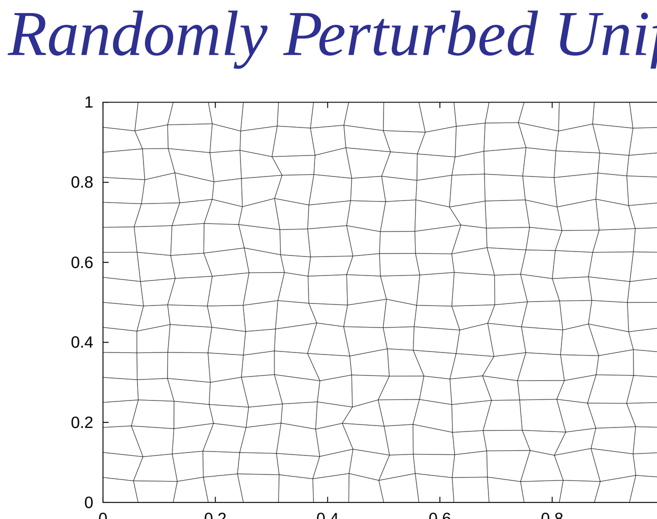
### *A Smooth Grid (global smooth mapping)*



Iteration Counts

| Mesh Size | $CG(\mathcal{S}_\phi)$ | $CG(\mathcal{S}^D_\phi)$ | $CG(\mathcal{S}^{TF}_\phi)$ |
|---|---|---|---|
| $16\times16$ | 43 | 9 | 4 |
| $32\times32$ | 85 | 10 | 5 |
| $64\times64$ | 164 | 10 | 5 |
| $128\times128$ | 319 | 10 | 5 |

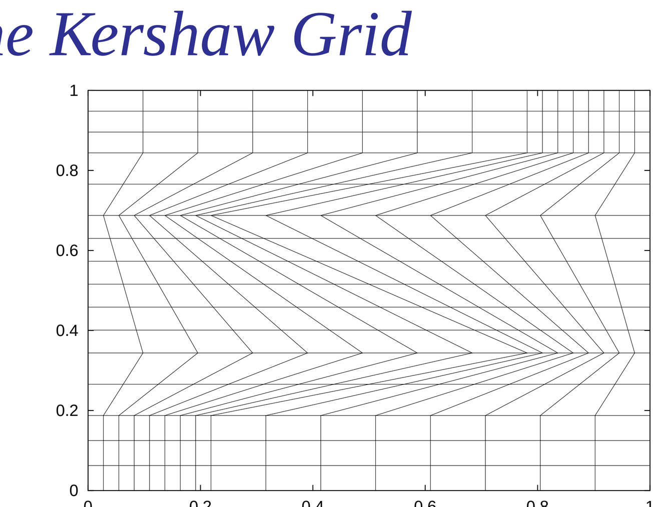
### *A Randomly Perturbed Uniform Grid*



Iteration Counts

| Mesh Size | $CG(\mathcal{S}_\phi)$ | $CG(\mathcal{S}^D_\phi)$ | $CG(\mathcal{S}^{TF}_\phi)$ |
|---|---|---|---|
| $16\times16$ | 43 | 8 | 4 |
| $32\times32$ | 78 | 9 | 4 |
| $64\times64$ | 144 | 9 | 4 |
| $128\times128$ | 279 | 9 | 4 |

### *The Kershaw Grid*



Iteration Counts

| Mesh Size | $CG(\mathcal{S}_\phi)$ | $CG(\mathcal{S}^D_\phi)$ | $CG(\mathcal{S}^{TF}_\phi)$ |
|---|---|---|---|
| $16\times16$ | 53 | 34 | 15 |
| $32\times32$ | 108 | 44 | 15 |
| $64\times64$ | 211 | 52 | 14 |
| $128\times128$ | 406 | 58 | 13 |

## Conclusions

*PCG on the reduced scalar system*

- ➢ augmented system leads to a 9-point cell-based preconditioner
  - ① robust with respect to grid distortion
  - ② convergence is h-independent
  - ③ readily extends to three dimensions
- ➢ inversion of $\mathcal{A}$ needs to be done efficiently.

*Avoiding the inversion of $\mathcal{A}$*

- ➢ Preconditioned GMRES on the full system
- ➢ Preconditioned CG on the reduced scalar system of the *local* SOM.

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